Chapter 3
Time dilation from the classical wave equation
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from my book:
Understanding Relativistic Quantum Field Theory

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Time dilation from the classical wave equation

3.1 Signal propagation: The bouncing photon clock

The classical wave equation tells us that the propagation is on the light cone, and the propagation speed is c. With this as a starting point we will show that we should expect that physical processes which move progress slower as they do when at rest. In this chapter we will consider Time dilation from a single reference point of view before we will handle the emergence of non-simultaneity from the classical wave equation in the next chapter.

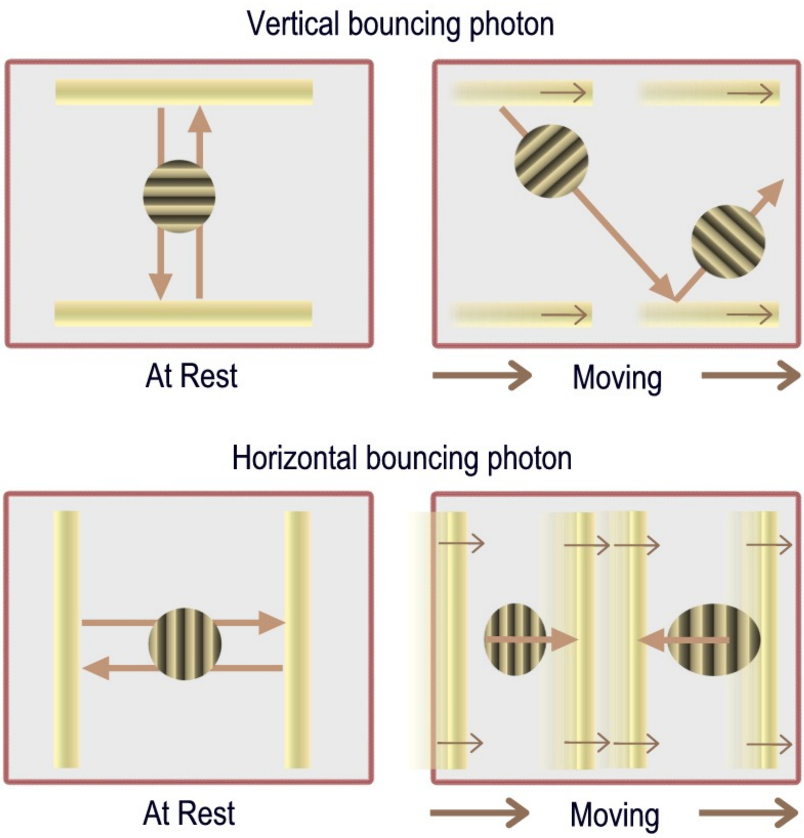

Figure 3.1: Bouncing photon clock, at rest (left), moving (right)

One of the simplest clocks one can conceive is that of a single photon bouncing between two mirrors. This elementary timing device also symbolizes the maximum rate of how fast two objects will interact, or more general, how fast physical processes do progress.

The clock-rate becomes higher if the distance between the two mirrors decreases, and slower if the distance increases. The interaction between objects proceeds faster or slower depending on the distance between them.

Figure 3.1 shows vertically and horizontally bouncing photons in the rest-frame as well as in a moving frame. We will see that the analysing the clock just amounts to a review of the classical Michelson and Morley experiment.

For a clock with the photon bouncing vertical (orthogonal to the velocity) we see that the "tick" and the "tock" are equal in duration. The photon moves on the diagonals with the speed of light c while the mirrors move horizontally with speed v. The vertical component of the speed which determines the duration of the ticks is thus $\sqrt{c^2 - v^2}$ and the duration of the ticks for a distance L between the mirrors becomes.

$$T_{tick} \;=\; T_{tock} \;=\; \frac{L}{\sqrt{c^2 - v^2}} \;=\; \gamma\frac{L}{c} \tag{3.1}$$

In the case, where the photon bounces horizontal we get an asymmetry. The photon moving along with the mirrors in the same direction takes more time to go from one mirror to the other as the photon moving in the opposite direction as the mirrors. The times T_{tick} and T_{tock} are different.

$$T_{tick} + T_{tock} \;=\; \frac{(L/\gamma)}{c - v} + \frac{(L/\gamma)}{c + v} \;=\; 2\,\frac{(L/\gamma)}{c^2 - v^2} \;=\; 2\,\gamma\frac{L}{c} \tag{3.2}$$

However, in both cases the total time for the tick plus the tock is $2\gamma L/c$, compared with a total time of $2L/c$ for a clock at rest. In both cases the clock runs slower by a factor γ. The factor γ which determines the time dilation.

Closer observation of figure 3.1 shows that the wave length of light changes when we look at it from the moving frame. The lower right image shows the Doppler effect on the photons bouncing of the mirrors. The wave length becomes shorter if a photon is reflected by the mirror moving towards it, while the wavelength becomes longer if it's reflected by the other mirror.

Even more interesting are the diagonal wavefronts in the top right image of figure 3.1. The wave fronts have rotated in such a way that they point in the direction of the propagation, as they should do.

These transformations observed by going from one reference frame to another go beyond simple Lorentz contraction and are of course the result of the change in simultaneity. We can read of the change in simultaneity most easily from the top right image with the diagonal wavefronts:

The time changes in the horizontal direction (the zones of equal time are vertical bands). Looking at the wavefront we can see that time is further advanced at the left side (Δt is positive), because that part of the wavefront has propagated closer towards the opposite mirror. The time at the right side is trailing (Δt is negative), shown by the trailing wavefront. This skew in the propagation of the wavefront is what causes it to rotate.

Non-simultaneity is also what causes the Doppler effects in the bottom right image. We will explain how non-simultaneity occurs physically (and how it can be predicted from classical physics) in the next chapter which is devoted specificality to non-simultaneity.

3.2 Twin Brothers in a single reference frame.

A good way to physically understand the seemingly paradoxical aspects of time dilation is to first review the twin "paradox" it in a single reference frame, without switching from one reference frame to another we avoid the extra complexities associated with non-simultaneity.

Figure 3.2 shows the cases we will calculate here. We will see that in both cases it is the traveling twin brother who ages less. The stripes on the lines are an indication of the clock ticks.

Aging goes fastest in this reference frame for an object at rest and everything what moves progresses slower. The faster it moves, the slower it progresses. Now look at the two cases of the Twin "paradox"

The left side image shows twin brother B traveling That is, he moves away from his brother A for some time and changes speed at half the travel time (half of his time) to turn back to his twin brother A. It's evidently that twin brother B ages less, OK, but now the other way around

The right side image starts the same initially but now it's twin brother A who is traveling. His distance to B is increasing first but then, after half of (his) travel time, he changes speed in order to meet up with his twin brother B at the end of the voyage.

To do so he has to go after twin brother A at a higher speed in order to catch up with him at the end of the voyage. At this point they can compare their ages.

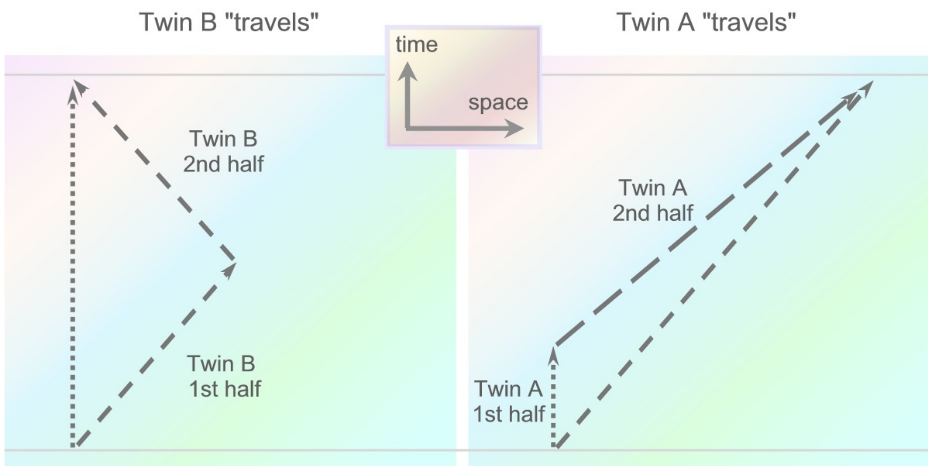

Figure 3.2: The twin brothers "paradox" in a fixed reference frame

It turns out that the situation is reversed now. It's the first of the twins who has aged less. (by exactly the same amount). This is because of the higher speed the first twin needs to travel to overtake his brother. Aging slows down so much (in this shorter period) that at the end he has aged less as his brother who continued to travel always at the same speed.

Now look at case 2 from the eyes of the second brother: He first sees his brother getting farther and farther away and then, somewhere halfway, he sees his brother changing speed to get back to him.

The second brother may presume that he's at rest all the time and that it's the first twin brother who went on a voyage and came back, and, therefore the first brother should have aged less, which is indeed exactly what they find out!

We can use the standard formula for relativistic velocity "addition" to determine the speed which twin brother A needs to catch up.

$$v_{(a+b)} = \frac{v_a + v_b}{1 + v_a v_b / c^2} \tag{3.3}$$

Twin brother B, moving himself with v sees, in his reference frame, that twin brother A is catching up with him with the same speed v. The catch up speed as seen from our frame is therefor.

$$v_2 = \frac{2v}{1 + v^2/c^2} \tag{3.4}$$

The time dilation at this velocity v_2 is determined by γ_2 which is.

$$\gamma_2 = \frac{1}{\sqrt{1 - v_2^2/c^2}} = \frac{1 + v^2/c^2}{1 - v^2/c^2} \tag{3.5}$$

The time, in our frame, needed for twin brother A to catch up is given by.

$$\Delta T_2 = \frac{L}{v_2} = \frac{T}{2}\left(1 + \frac{v^2}{c^2}\right) \tag{3.6}$$

Where T is the total time. Consequently the first half of twin brother A's voyage, took a time of $\Delta T_1 = T - \Delta T_2$ in our reference frame, where he was at rest.

$$\Delta T_1 = T - \Delta T_2 = \frac{T}{2}\left(1 - \frac{v^2}{c^2}\right) \tag{3.7}$$

The first and second half should take equal amounts of proper time for twin brother A. If we apply the time dilation at a speed v_2 to determine the proper time $\Delta T_2'$ for twin brother A then we obtain indeed.

$$\Delta T_2' = \frac{T_2}{\gamma_2} = \frac{T}{2}\left(1 - \frac{v^2}{c^2}\right) \tag{3.8}$$

The proper times spend at the first and second half is equal, $\Delta T_1 = \Delta T_2'$

The total proper time for twin brother A is just the sum of the two.

$$T_A \quad = \quad \Delta T_1 + \Delta T_2' \quad = \quad T\left(1 - \frac{v^2}{c^2}\right) \quad = \quad \frac{T}{\gamma^2} \tag{3.9}$$

So we see that the time dilation for twin brother A is given by γ^2 versus γ for twin brother B who was moving at a constant speed. The traveling twin brother A has aged less by a factor γ compared with his twin brother who did not change velocity.

$$\begin{aligned} &\text{Time passed for A:} \qquad T_A \quad = \quad \tfrac{T}{\gamma^2} \quad = \quad T\left(1 - \tfrac{v^2}{c^2}\right) \\ &\text{Time passed for B:} \qquad T_B \quad = \quad \tfrac{T}{\gamma} \quad = \quad T\sqrt{1 - \tfrac{v^2}{c^2}} \end{aligned} \tag{3.10}$$

We see that we can deduce the slower aging of the traveling twin brother in both cases from the on-the-lightcone propagation associated with the 1+3d classical wave equation. We have not discussed non-simultaneity yet. The relativistic effect of non simultaneity, which can also be deduced from the classical wave equation, will make the two different cases discussed here 100% symmetric.

Until sofar we've only discussed the wave equation without mass term which covers the important class of electromagnetic interactions. In the following sections we will have an introductory discussion of the relativistic effect of time dilation in matter wave functions for particles with mass.

3.3 Nature's own clock: The de Broglie frequency

Nature has it's own perfect clocks in the de Broglie frequencies of objects with mass. The frequency in the rest frame is in principle what determines the invariant mass of particles. It is this frequency which determines how a particle propagates. In a moving reference frame this internal clock reacts in the same way as our bouncing photon clock.

Figure 3.3 shows a Minkowski diagram with de Broglie waves. Looking at the moving frame at the right along the trajectory of the particle (over the t'-axis) we see that the frequency reduces (by a factor γ) as defined by the time dilation.

Looking at the same image but now along the vertical axis (the t-axis) then we see that the frequency increases by a factor γ instead. This of course represents the energy dependence on the velocity $E = \gamma mc$. It is the effect of non-simultaneity, which causes the x'-axis to rate, that is responsible for this reversed effect.

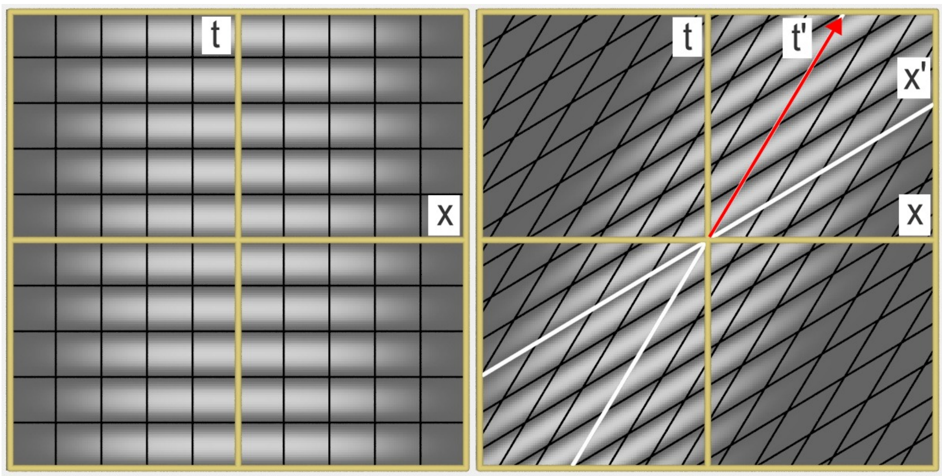

Figure 3.3: de Broglie waves of a particle at rest (left) and moving (right)

We have as yet still no descriptive model which tells us exactly what kind of microscopic movement causes the de Broglie frequency, but the fact that it transforms just as our bouncing photon clock strongly suggest that a motion through real space is involved here, even though one often sees the term "internal space" used. The latter term merely describes our ignorance about what is really going on.

Part of our ignorance stems from the fact that the effect of Time dilation is independent of the direction of the internal motion relative to the direction of motion of the object itself. We can at this stage not tell if we deal with an internal vibration or with an internal rotation or something else.

We will see later on that the Dirac equation strongly suggest that we have to do with some sort of rotation. In the next section however we will content ourself with the simplest model, that of an internal vibration, to handle the Wave equation for particles with mass.

3.4 The Wave equation for particles with mass

Our in depth treatment of massive particles with a de Broglie frequency comes in chapters on the Klein Gordon equation. This equation can be viewed as the classical Wave equation with an extra mass term.

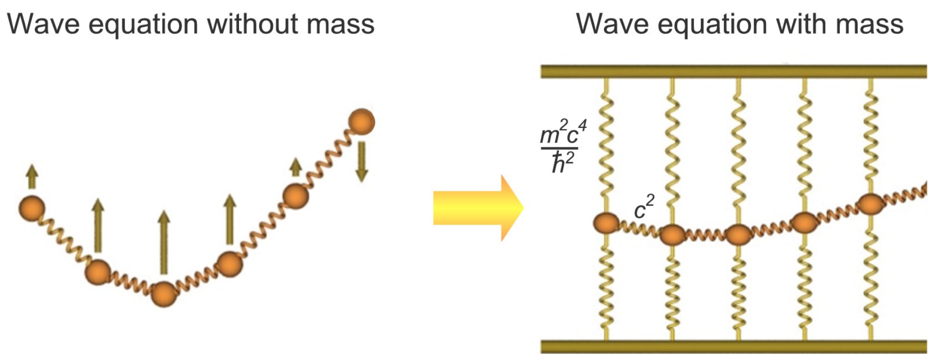

Figure 3.4: Wave equation for de Broglie waves

The mass term can be interpreted for instance in our mechanical equivalent by springs which oppose the perturbation ψ of the weights in the vertical direction. The constant $m^2c^2/\hbar^2$ represents the strength of the vertical springs. The minus sign at the right hand side indicates that the force is opposing the displacement.

$$\text{Klein Gordon equation:} \qquad \frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} = -\left(\frac{m^2c^4}{\hbar^2}\right)\psi \qquad (3.11)$$

The real eigenfunctions of this differential equation are given by.

$$\psi = \sin\left(-\tfrac{E}{\hbar}\,t + \tfrac{p}{\hbar}\,x\right), \qquad \psi = \cos\left(-\tfrac{E}{\hbar}\,t + \tfrac{p}{\hbar}\,x\right) \qquad (3.12)$$

Inserting the eigenfunctions into equation (3.11) gives us the classical energy- momentum relation.

$$-\left(\,E^2 - p^2c^2 - m^2c^4\,\right)\frac{\psi}{\hbar^2} = 0 \qquad (3.13)$$

Where the classical relativistic energy E and momentum p are given by.

$$E = \gamma m c^2, \qquad p = \gamma m v \tag{3.14}$$

The frequency in time is given by $f = E/h$. If there is no second derivative in x, which is only physically possible if ψ is the same everywhere, then E is given by the rest-mass energy $f = mc^2/h$ and the particle is at rest.

To check the Time dilation of the particle we look at the eigenfrequency of the particle along its trajectory which is given by $x = vt$, substituting this in the eigenfunction gives us.

$$\begin{aligned} \psi &= \sin\Big(- \left(\tfrac{E}{\hbar} - \tfrac{vp}{\hbar}\right) t \Big) \\ &= \sin\Big(- \tfrac{\gamma m}{\hbar} \left(c^2 - v^2\right) t \Big) \\ &= \sin\Big(- \tfrac{mc^2}{\gamma \hbar}\, t \Big) \end{aligned} \tag{3.15}$$

Which shows that the frequency of the moving particle along its trajectory becomes *lower* by a factor γ as it should according to the time dilation.

$$f = \frac{mc^2}{\gamma\, h} \tag{3.16}$$

This is the self oscillatory frequency of the particle. The eigenfunctions as defined in (3.12) can be considered as wave functions moving with velocity v *plus* this self oscillation. For every arbitrary shaped wave function φ which shifts along with a constant velocity v the following relation holds.

$$\frac{\partial \varphi}{\partial t} = -\, v\, \frac{\partial \varphi}{\partial x} \tag{3.17}$$

This relations should hold if we remove the self oscillation from the eigenfunction.

$$\begin{aligned} \varphi &= \sin\Big(- \Big(\tfrac{E}{\hbar} - \tfrac{mc^2}{\gamma \hbar}\Big)\, t + \tfrac{p}{\hbar} x \Big) \\ &= \sin\Big(- \tfrac{\gamma m}{\hbar} \left(v^2 t - v x\right) \Big) \end{aligned} \tag{3.18}$$

Which indeed describes a function shifting along with v according to (3.17)

3.5 Decomposition to lightcone propagators

We derived the time dilation from our "bouncing photon clock" which, in the specific case of 3 spatial dimensions, is governed by a propagator which propagates uniquely on the lightcone.

The propagator for massive particles does, obviously, not propagate on the lightcone, because a massive particle can have any speed between 0 and c. We can however decompose the propagator into a series of pure lightcone propagators. Defining $\Box$ as the $1+3d$ dimensional d'Alembertian.

$$\Box \quad = \quad \left(\frac{\partial^2\psi}{\partial t^2} \;-\; c^2\frac{\partial^2\psi}{\partial x^2} \;-\; c^2\frac{\partial^2\psi}{\partial y^2} \;-\; c^2\frac{\partial^2\psi}{\partial z^2} \right) \tag{3.19}$$

and simplifying the notation a bit by using natural units, $\hbar = c = 1$, we can express the Klein Gordon equation as.

$$\Box\psi \quad = \quad -\, m^2\psi \tag{3.20}$$

Which shows that we can consider the term $-m^2\psi$, representing the field itself, as a source. The field which originates from this term opposes the existing field, just like the vertical strings oppose the displacement of the masses in figure 3.4

To obtain the propagator of the wave equation for particles with mass we need to invert the operator.

$$\left(\Box + m^2\right)\psi \quad = \quad \delta(t,x,y,x) \tag{3.21}$$

The field ψ is the result of the delta function perturbation. We are looking for the inverse operator which acts on the delta function to give the field ψ. This inverse operator is given by.

$$\left(\Box + m^2\right)^{-1} \quad = \quad \Box^{-1} \;-\; m^2\,\Box^{-2} \;+\; m^4\,\Box^{-3} \;-\; m^6\,\Box^{-4} \;+\; .... \tag{3.22}$$

The first term $\Box^{-1}$ at the right is the same as in the massless case ($m = 0$). It represents the propagation from a delta source on the lightcone. The second term is the first re-propagation. Each point of the field from the first term acts itself as a new source with a negative sign, tending to suppress the field. The third term is the second re-propagation, and so on.

All the re-propagating terms propagate on the lightcone. The sum, the total propagator, is not on the lightcone anymore because each point acting as a new source propagates in all directions including the directions opposite to the original propagation.

Nevertheless we can derive the time dilation of special relativity from each of the lightcone propagators, as well as from the sum of the time-ordered products of lightcone propagators.

Fourier domain representation

We can, since the eigenfunctions are sinusoidal, look at the fourier domain where the operators become algebraic expressions of the eigenvalues. The Fourier transform of the d'Alembert operator is.

$$\mathcal{F}\{\Box\} \;=\; -\,q^2 \;=\; -\,E^2 + p_x^2 + p_y^2 + p_z^2 \tag{3.23}$$

The Fourier transform of the inverse operator $(\Box + m^2)^{-1}$ can be expressed as the following series.

$$-\Big(q^2 - m^2\Big)^{-1} \;=\; -\Bigg(\frac{1}{(q^2)} + \frac{m^2}{(q^2)^2} + \frac{m^4}{(q^2)^3} + \frac{m^6}{(q^2)^4} + \,.....\Bigg) \tag{3.24}$$

These terms correspond one to one with the terms of equation (3.22)